ÖAW IWF

Cassini states in the spin-orbit problem

CELMEC VII

Viterbo, Italy, 04 / 09 / 2017

Christoph Lhotka

Space Research Institute, Austrian Academy of Sciences

ÖAW IWF

Motivation of the study

Aims :

- Investigate the effect of higher order spin-orbit resonances on the orientation of the spin-axis of a rigid body
- Investigate the effect of higher degree gravity harmonics on the spin-frequency and orientation of the spin axis
- Understand the effect of the core and flattening of a celestial body on possible orientations of the spin axis of Mercury in the past

Outlook :

- Understand long-term stability of spin-orbit coupling and orientation of the spin-axis

Results :

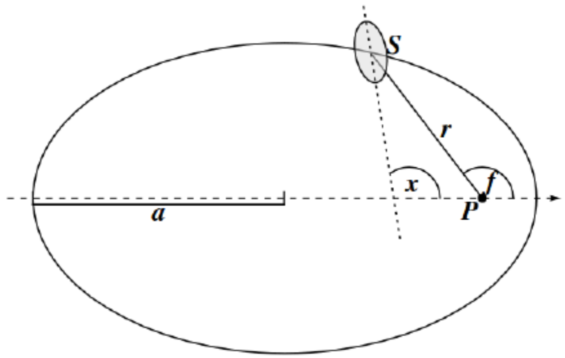
- Set-up basic resonant models for the investigation of the long-term stability of coupled spin-orbit problems
- We provide a systematic study of p:q spin-orbit resonances with $p,q \leq 8$
- Include gravitational potential of the rotator up to degree and order 4
- Quasi linear relation ship between order of the resonances and flattening of Mercury are found

The presentation summarizes recent results accepted for publication in CM&DA, Lhotka, 2017

ÖAW IWF

Spin-orbit problem & Cassini states

Spin-orbit problem :



- Describes both the orbital and rotational motion of celestial bodies
- Includes the coupling between orbital dynamics and rotational dynamics
- Find special solutions of the problem, investigate the geometry and long-term stability of motions

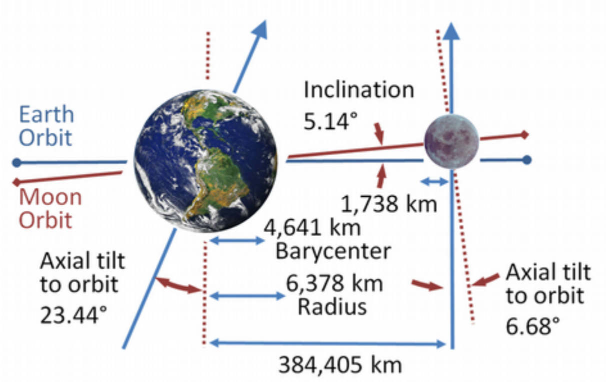
Cassini laws (for the Moon)

1) The rotation rate is synchronous with the mean orbital rate

2) The spin axis maintains a constant inclination to the orbital plane

3) The spin axis, orbit normal, and ecliptic normal remain coplanar

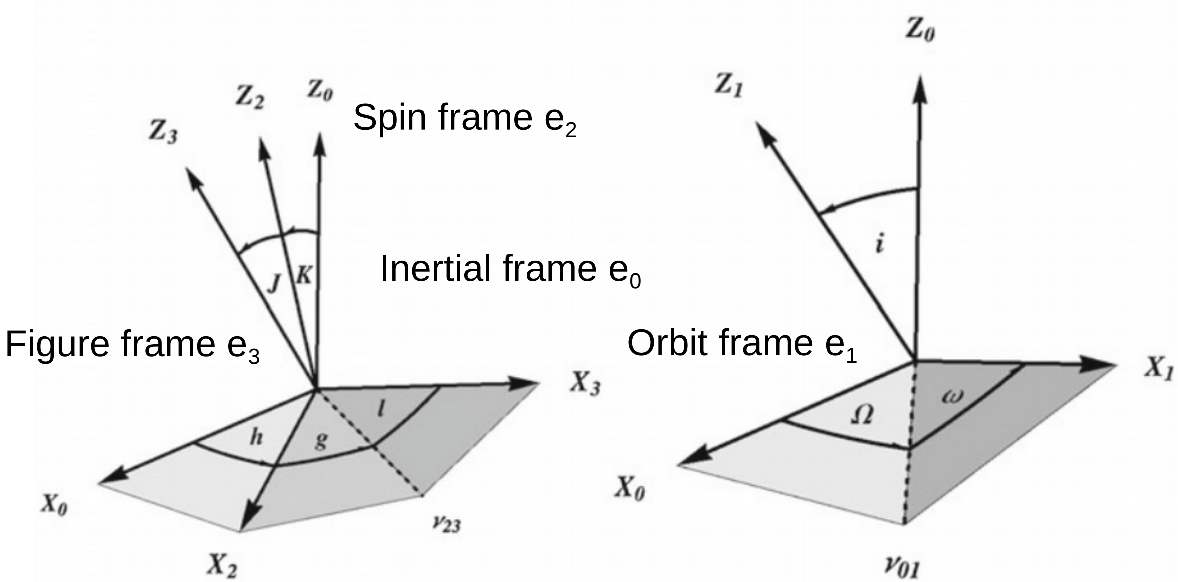
Cassini states :



- Cassini states correspond to equilibria of the orientation axis in the spin-orbit problem
- Constant inclination of rotation axis (Law 2)
- Coupling between the ascending node longitude and node of the spin-frame (Law 3)

ÖAW IWF

Notation, reference frames & variables



Rotational dynamics

Symbol	Name
$l^{(o)}=l$	spin angle
$g^{(o)}=g$	Figure frame angle
$h^{(o)}=h$	spin frame angle
J	wobble
K	inertial obliquity

Orbital dynamics

Symbol	Name
a	Semi-major axis
e	Eccentricity
i	Inclination
$g^{(o)}=\omega$	Pericenter argument
$h^{(o)}=\Omega$	Ascending node lon.
$l^{(o)}=M$	Mean anomaly

$$\mu = G_c(M+m)$$

Modified Andoyer variables :

$$L_1 = G^{(s)}, \qquad l_1 = l^{(s)} + g^{(s)} + h^{(s)},$$
$$L_2 = G^{(s)} - L^{(s)} = G^{(s)}(1-\cos(J)), \qquad l_2 = -l^{(s)},$$
$$L_3 = G^{(s)} - H^{(s)} = G^{(s)}(1-\cos(K)), \qquad l_3 = -h^{(s)}.$$

Modified Delaunay variables :

$$L_4 = L^{(o)} = \sqrt{\mu a}, \qquad l_4 = l^{(o)} + g^{(o)} + h^{(o)},$$
$$L_5 = L^{(o)} - G^{(o)} = \sqrt{\mu a}\left(1-\sqrt{1-e^2}\right), \qquad l_5 = -g^{(o)} - h^{(o)},$$
$$L_6 = G^{(o)} - H^{(o)} = \sqrt{\mu a\left(1-e^2\right)}(1-\cos(i)), \; l_6 = -h^{(o)}.$$

ÖAW IWF

The dynamical problem

$$\mathcal{H} = \boxed{\mathcal{H}_s} + \boxed{\mathcal{H}_o} + \boxed{\mathcal{V}_r}$$

Hamiltonian function

Spin $\frac{1}{2}\left(L_1{}^2 - (L_1 - L_2)^2\right)\left(\frac{\sin(l_2)^2}{A} + \frac{\cos(l_2)^2}{B}\right) + \frac{(L_1 - L_2)^2}{2C}$ ...Andoyer problem

Orbit $-\frac{m^3\mu^2}{2L_4{}^2} + \dot{\omega}L_5 + \dot{\Omega}L_6$ … (perturbed) Kepler problem

Problem $-\frac{\mathcal{GM}}{r}\sum_{n=2}^{\infty}\sum_{m=0}^{n}\left(\frac{R}{r}\right)^n P_{nm}(\sin(\phi))\left(C_{nm}\cos(m\lambda) + S_{nm}\sin(m\lambda)\right)$ ...Gravitational coupling

Rotation of celestial body is coupled to its orbital motion with respect to other celestial bodies.

We seek to find special kinds of motion, i.e. repeating patterns, in the evolution in time of the dynamical system :

Action - angle variables :
Delaunay & Andoyer variables

$$\begin{aligned}
\Sigma_1 &= L_1 , & \sigma_1 &= l_1 - \tfrac{p}{q} l_4 - l_5 - l_6 , \\
\Sigma_2 &= L_2 , & \sigma_2 &= l_2 , \\
\Sigma_3 &= L_3 , & \sigma_3 &= l_3 + l_6 , \\
\Sigma_4 &= L_4 + \tfrac{p}{q} L_1 , & \sigma_4 &= l_4 , \\
\Sigma_5 &= L_5 + L_1 , & \sigma_5 &= l_5 , \\
\Sigma_6 &= L_6 + L_1 - L_3 , & \sigma_6 &= l_6 .
\end{aligned}$$

Resonant variables that define the coupling for the p:q resonance.

Gravitational coupling

Start : Spherical harmonic expansion of gravitational potential of the rotator :

Legendre polynomials

Gravitational mass parameter

Spherical harmonic coefficients

$$\mathcal{V} = -\frac{\mathcal{GM}}{r}\sum_{n=2}^{\infty}\sum_{m=0}^{n}\left(\frac{R}{r}\right)^n P_{nm}(\sin(\phi))\,(C_{nm}\cos(m\lambda) + S_{nm}\sin(m\lambda))$$

distance from the center of mass

Spherical coordinates

Aim : Isolation of dominant terms in resonant action-angle variables :

p:q resonance

$$\mathcal{V}_r = -\mathcal{GM}\sum_{k_1,k_2}^{\infty} c_{k_1,k_2}\cos\left(k_1\sigma_1 + k_3\sigma_3\right) + s_{k_1,k_2}\sin\left(k_1\sigma_1 + k_3\sigma_3\right)$$

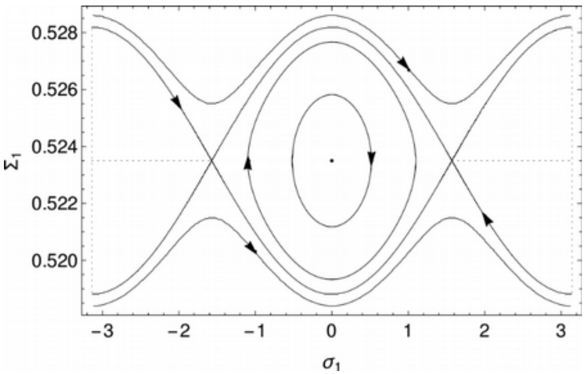
Keep terms of the form : $k_1\sigma_1 + k_3\sigma_3 = k_1 l_1 + k_3 l_3 - k_1\frac{p}{q}l_4 - k_1 l_5 + (k_3 - k_1)\,l_6$

Neglect terms of the form : $k_1\sigma_1 + k_3\sigma_3 + k_4 l_4 + k_5 l_5 + k_6 l_6$ with $qk_4 \neq -k_1 p,\ k_5 \neq -k_1,\ k_6 \neq k_3 - k_1$

ÖAW IWF

Derivation of the resonant model

Spin frequency and mean motion:

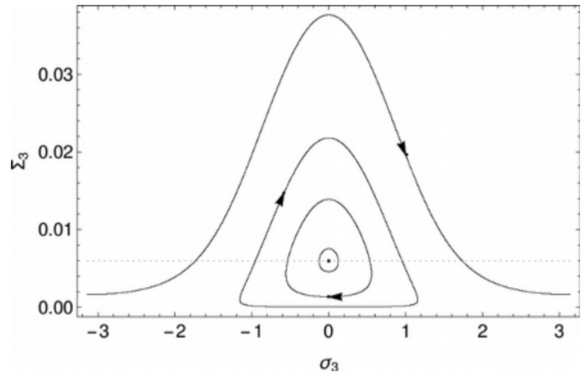


$$\dot{l}_1 = \frac{L_1}{C} \equiv n_s \ , \qquad \dot{l}_4 = \frac{m^3 \mu^2}{L_4^3} = n$$

$$\frac{T_s}{T} = \frac{p}{q} \ , \quad \text{with} \quad \mathrm{p, q} \in \mathbb{N}_+ / \{0\}$$

$$q n_s = p n$$

Coupling between nodal motions:



$$\dot{l}_3 - \dot{l}_6 = 0$$

$$\frac{T_h}{T_\Omega} = 1$$

Transformation to resonant variables:

$$S_{p:q} = \Sigma_1 \left(l_1 - \frac{p}{q} l_4 - l_5 - l_6 \right) + \Sigma_2 l_2 + \Sigma_3 \left(l_3 + l_6 \right) + \Sigma_4 l_4 + \Sigma_5 l_5 + \Sigma_6 l_6$$

4-dimensional phase space :

- (Σ_1,σ_1) plane : spin-orbit resonances
- (Σ_3,σ_3) plane : Cassini states

Spin-orbit dynamical system:

$$\dot{\sigma} = \frac{\partial \mathcal{H}}{\partial \Sigma} \ , \qquad \dot{\Sigma} = -\frac{\partial \mathcal{H}}{\partial \sigma}$$

ÖAW IWF

Expansion of the gravitational field I

$$\mathcal{V} = -\frac{\mathcal{GM}}{r}\sum_{n=2}^{\infty}\sum_{m=0}^{n}\left(\frac{R}{r}\right)^n P_{nm}(\sin(\phi))\left(C_{nm}\cos(m\lambda)+S_{nm}\sin(m\lambda)\right)$$

- Calculations performed using specialized Computer Algebra Routines
- Poisson Series Processor handling rational polynomial coefficients and trigonometric terms
- Write out of intermediate steps of the calculation to check validity of calculations
- Implemented possible on a workstation PC with 8 GB of RAM
- Variable degree and order in gravity harmonics and order in eccentricity and inclination
- Calculations up to 1) harmonic degree and order 4 (up to C_{44} and S_{44}), 2) order 4 in eccentricity
- Unaveraged potential contains 7.414.883 Fourier / polynomial terms

available soon at https://l-sgn.org/cmda-2017/

We investigated p:q resonances with p>q and p,q=1,...8

First Fourier order term that remains after average of the resonant model

Only a few p:q resonances are complete up to $O(e^5)$*

	C_{11}, S_{11}	C_{20}	C_{22}, S_{22}	C_{31}, S_{31}	C_{33}, S_{33}	C_{40}	C_{42}, S_{42}	C_{44}, S_{44}
3 : 1	σ_1	σ_3	$2\sigma_1$	σ_1	–	σ_3	$2\sigma_1$	$2\sigma_1$
5 : 2	–	σ_3	$2\sigma_1$	–	–	σ_3	$2\sigma_1$	$2\sigma_1$
2 : 1	σ_1	σ_3	$2\sigma_1$	σ_1	$3\sigma_1$	σ_3	$2\sigma_1$	$2\sigma_1$
3 : 2	–	σ_3	$2\sigma_1$	–	–	σ_3	$2\sigma_1$	$2\sigma_1$
1 : 1	σ_1	σ_3	$2\sigma_1$	σ_1	$3\sigma_1$	σ_3	$2\sigma_1$	$2\sigma_1$

* contain all resonant arguments

Expansion of the gravitational field II

Example : C_{20} is proportional to $\frac{R^2\left(3z^2-1\right)}{2r^3}$ in a bodycentric reference frame

- from $\mathcal{V} = -\frac{\mathcal{GM}}{r}\sum_{n=2}^{\infty}\sum_{m=0}^{n}\left(\frac{R}{r}\right)^n P_{nm}(\sin(\phi))\left(C_{nm}\cos(m\lambda)+S_{nm}\sin(m\lambda)\right)$
- transformation of the component $\frac{1}{2}\left(3\sin^2(\varphi)-1\right)$ to Cartesian variables $\frac{1}{2}\left(3z^2-1\right)$
- with $z = Zc_K - s_K\left(Yc_{l_3}+Xs_{l_3}\right)$ to inertial reference frame,
- and with

$$\begin{aligned} X &= c_{M+\omega}\left(c_\Omega - 2ec_i s_M s_\Omega\right) - s_{M+\omega}\left(2ec_\Omega s_M + c_i s_\Omega\right) \\ Y &= c_i c_\Omega\left(2es_M c_{M+\omega} + s_{M+\omega}\right) + s_\Omega\left(c_{M+\omega} - 2es_M s_{M+\omega}\right) \\ Z &= s_i\left(2es_M c_{M+\omega} + s_{M+\omega}\right) \end{aligned}$$

from orbital reference frame :

- gives

$$-\frac{R^2}{2a^3}-\frac{3eR^2c_{l_4}}{2a^3}-\frac{3eR^2c_{l_4}c_{l_5}^2c_{l_6}^2s_K^2s_{l_3}^2}{a^3}+\frac{3R^2c_{l_4}^2c_{l_5}^2c_{l_6}^2s_K^2s_{l_3}^2}{2a^3}+\frac{15eR^2c_{l_4}^3c_{l_5}^2c_{l_6}^2s_K^2s_{l_3}^2}{2a^3}$$

$$\frac{3eR^2c_Kc_{l_5}^2c_{l_6}s_is_Ks_{l_3}s_{l_4}}{a^3}-\frac{3R^2c_Kc_{l_4}c_{l_5}^2c_{l_6}s_is_Ks_{l_3}s_{l_4}}{a^3}-\frac{18eR^2c_Kc_{l_4}^2c_{l_5}^2c_{l_6}s_is_Ks_{l_3}s_{l_4}}{a^3}-\frac{3eR^2c_ic_{l_3}c_{l_5}^2c_{l_6}^2s_K^2s_{l_3}s_{l_4}}{a^3}$$

$$\frac{3R^2c_ic_{l_3}c_{l_4}c_{l_5}^2c_{l_6}^2s_K^2s_{l_3}s_{l_4}}{a^3}+\frac{18eR^2c_ic_{l_3}c_{l_4}^2c_{l_5}^2c_{l_6}^2s_K^2s_{l_3}s_{l_4}}{a^3}+\frac{3R^2c_K^2c_{l_5}^2s_i^2s_{l_4}^2}{2a^3}+\frac{21eR^2c_K^2c_{l_4}c_{l_5}^2s_i^2s_{l_4}^2}{2a^3}$$

...plus 157 terms at O(e^2), 340 terms at O(e^3), 586 terms at O(e^4), 922 terms at O(e^5),...

- gives $f_0(\Sigma)+f_1(\Sigma)\cos(\sigma_3)+f_2(\Sigma)\cos(2\sigma_3)+\ldots$ (just for C_{20})

(after trigonometric reduction, introduction of resonant variables & averaging)

Cassini state 1 : stable equilibrium

Cassini state 1 is given by the condition :

$$\sigma_1 = \sigma_3 = 0$$
$$\Sigma^* = (\Sigma_1^*, \Sigma_3^*)$$

Equilibrium actions are determined by :

$$f_1(\Sigma_1, \Sigma_3) \equiv \left(\frac{\partial \mathcal{H}}{\partial \Sigma_1}\right)_{\sigma_1, \sigma_3 = 0} = 0$$
$$f_3(\Sigma_1, \Sigma_3) \equiv \left(\frac{\partial \mathcal{H}}{\partial \Sigma_3}\right)_{\sigma_1, \sigma_3 = 0} = 0$$

Relation between resonant variables and physical quantities :

$$\Sigma_3^* = \Sigma_1^* (1 - \cos(K_*))$$
$$\Sigma_1^* = G_*^{(s)}$$
$$K_* = i - \varepsilon_*$$

Necessary change of angular momentum (spin-frequency) & obliquity (inclination between rotation axis and orbit normal)

	3:1	5:2	2:1
$A[C_{11}]$	$\frac{a}{R}$	$-$	$\frac{a}{R}$
$A[C_{20}]$	1	1	1
$A[C_{22}]$	1	1	1
$A[C_{40}]$	$\frac{R^2}{a^2}$	$\frac{R^2}{a^2}$	$\frac{R^2}{a^2}$
$s_{C_{11},1}$	$\frac{9e^2}{16} - \frac{9e^4}{16}$	$-$	$\frac{e}{2} - \frac{3e^3}{8}$
$s_{C_{20},2}$	$-\frac{15e^4}{32} - \frac{3e^2}{8} - \frac{1}{4}$	$-\frac{9e^4}{16} - \frac{9e^2}{20} - \frac{3}{10}$	$-\frac{45e^4}{64} - \frac{9e^2}{16} - \frac{3}{8}$
$s_{C_{22},1}$	$\frac{533e^4}{32}$	$\frac{169e^3}{16}$	$\frac{51e^2}{8} - \frac{115e^4}{8}$
$s_{C_{22},2}$	$\frac{533e^4}{64}$	$\frac{169e^3}{32}$	$\frac{51e^2}{16} - \frac{115e^4}{16}$
$s_{C_{40},2}$	$\frac{525e^4}{512} + \frac{25e^2}{64} + \frac{5}{64}$	$\frac{315e^4}{256} + \frac{15e^2}{32} + \frac{3}{32}$	$\frac{1575e^4}{1024} + \frac{75e^2}{128} + \frac{15}{128}$
$s_{C_{40},4}$	$\frac{3675e^4}{1024} + \frac{175e^2}{128} + \frac{35}{128}$	$\frac{2205e^4}{512} + \frac{105e^2}{64} + \frac{21}{64}$	$\frac{11025e^4}{2048} + \frac{525e^2}{256} + \frac{105}{256}$

Conditions for equilibria that corresponds to Cassini state 1 :

$$G^{(s)} = \frac{p}{q}\frac{n^2}{\dot{\Omega}\sin(i-\varepsilon)}\sum_k kA[k]\sum_j s_{kj}\sin(j\varepsilon)$$

$$c\left(1 + \frac{q}{p}\frac{\dot{\omega}}{n} + \frac{q}{p}\frac{\dot{\Omega}}{n}\cos(i-\varepsilon)\right) = \frac{n}{\dot{\Omega}\sin(i-\varepsilon)}\sum_k kA[k]\sum_j s_{kj}\sin(j\varepsilon)$$

ÖAW IWF

Libration widths & fundamental periods

The resonant mathematical model takes the form :

$$\alpha\Sigma_1 + \alpha'\Sigma_3 + \frac{\beta}{2}\Sigma_1^2 + \gamma\cos(2\sigma_1) + \gamma'\cos(\sigma_3)\ldots$$

The maximum libration width can be bounded by :

$$\left(4\gamma\beta^{-1}\right)^{1/2}$$

Polynomial expansion reveals the quadratic term :

$$\mathcal{H}_2 = \gamma_{\Sigma_1\Sigma_1}\Sigma_1^2 + \gamma_{\Sigma_1\Sigma_3}\Sigma_1\sigma_3 + \gamma_{\Sigma_3\Sigma_3}\Sigma_3^2 + \gamma_{\sigma_1\sigma_1}\sigma_1^2 + \gamma_{\sigma_1\sigma_3}\sigma_1\sigma_3 + \gamma_{\sigma_3\sigma_3}\sigma_3^2$$

Introduction of local action-angle variables :

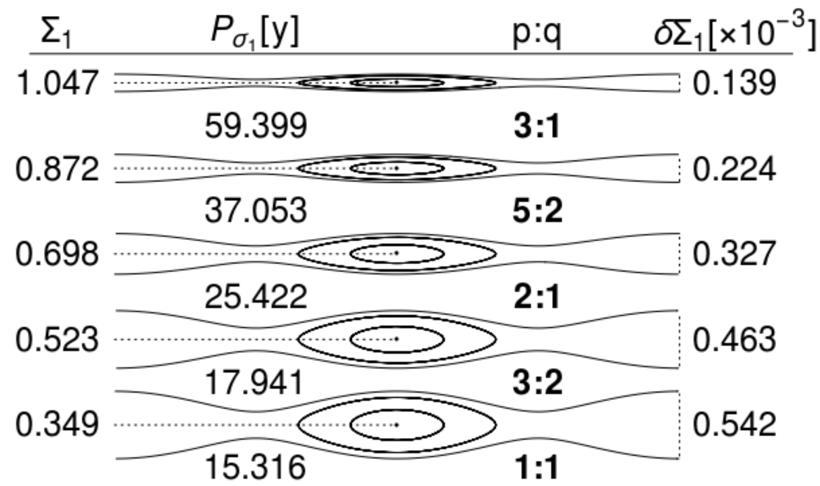
$$\sigma_1' = \sqrt{2UU_s}\sin(u)\ ,\quad \Sigma_1' = \sqrt{2U/U_s}\cos(u)$$
$$\sigma_3' = \sqrt{2VV_s}\sin(v)\ ,\quad \Sigma_3' = \sqrt{2V/V_s}\cos(v)$$

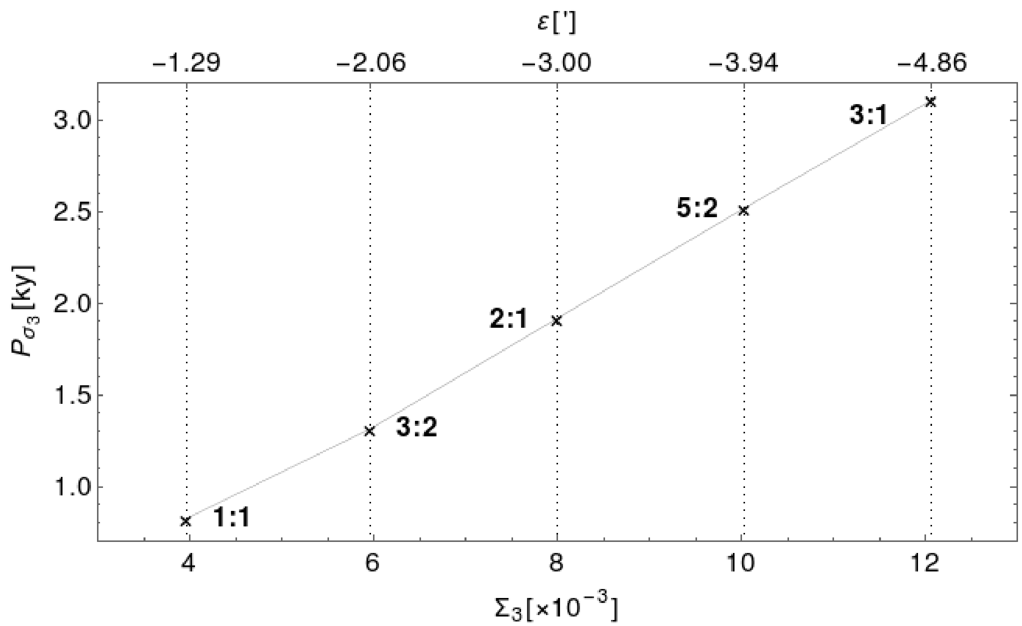
Reveals fundamental periods :

$$\mathcal{H}_2(U,V) = \omega_1 U + \omega_3 V$$

Reveals fundamental periods :

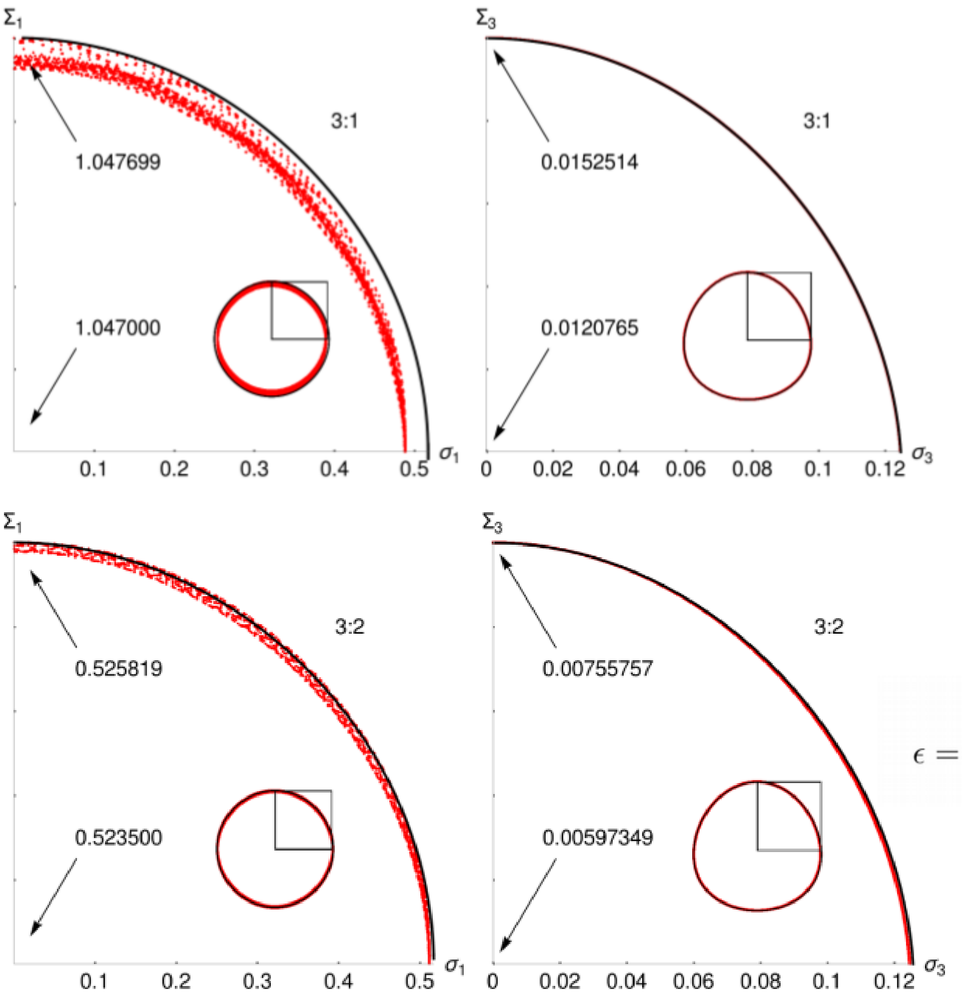
$$P_{\sigma_1} = 2\pi/\omega_1,\ P_{\sigma_3} = 2\pi/\omega_3$$





ÖAW IWF

Validation of the averaged model



- Comparison of averaged vs. unaveraged dynamical problem shows slight deviations due to :
- Non-resonant terms induce additional frequencies and oscillations in the full model
- Initial conditions are the same in the averaged and unaveraged problem

Checked agreement with Peale's alike formulas :

Yseboodt & Margot, 2006

$$\epsilon = -\frac{C\dot{\Omega}\sin i}{C\dot{\Omega}\cos i + 2nmR^2\left(\frac{7}{2}e - \frac{123}{16}e^3\right)C_{22} - nmR^2\left(1-e^2\right)^{-3/2}C_{20}}$$

Sansottera et al., 2015

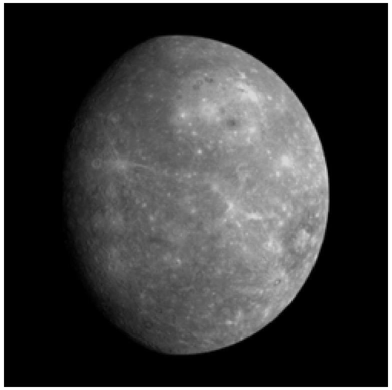
$$c = \frac{n\sin(\varepsilon)\left(C_{20}H_{20}\cos(\varepsilon) + C_{22}H_{22}(\cos(\varepsilon)+1)\right)}{\dot{\Omega}\sin(i-\varepsilon)\left(\frac{2\dot{\Omega}\cos(i-\varepsilon)}{3n} + \frac{2\dot{\omega}}{3n} + 1\right)}$$

Realizations in the solar system I

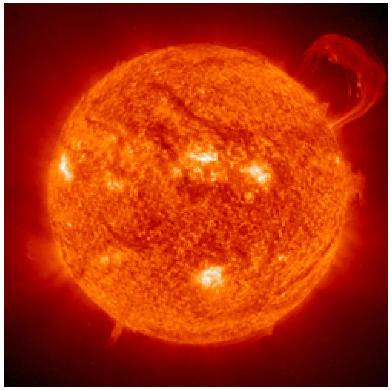
Our Moon :

around the Earth :

Planet Mercury :



around the Sun :



Moon Deimos :

around Mars :

Minor planet Charon :

around Pluto :

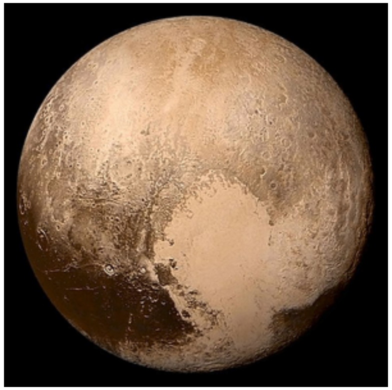


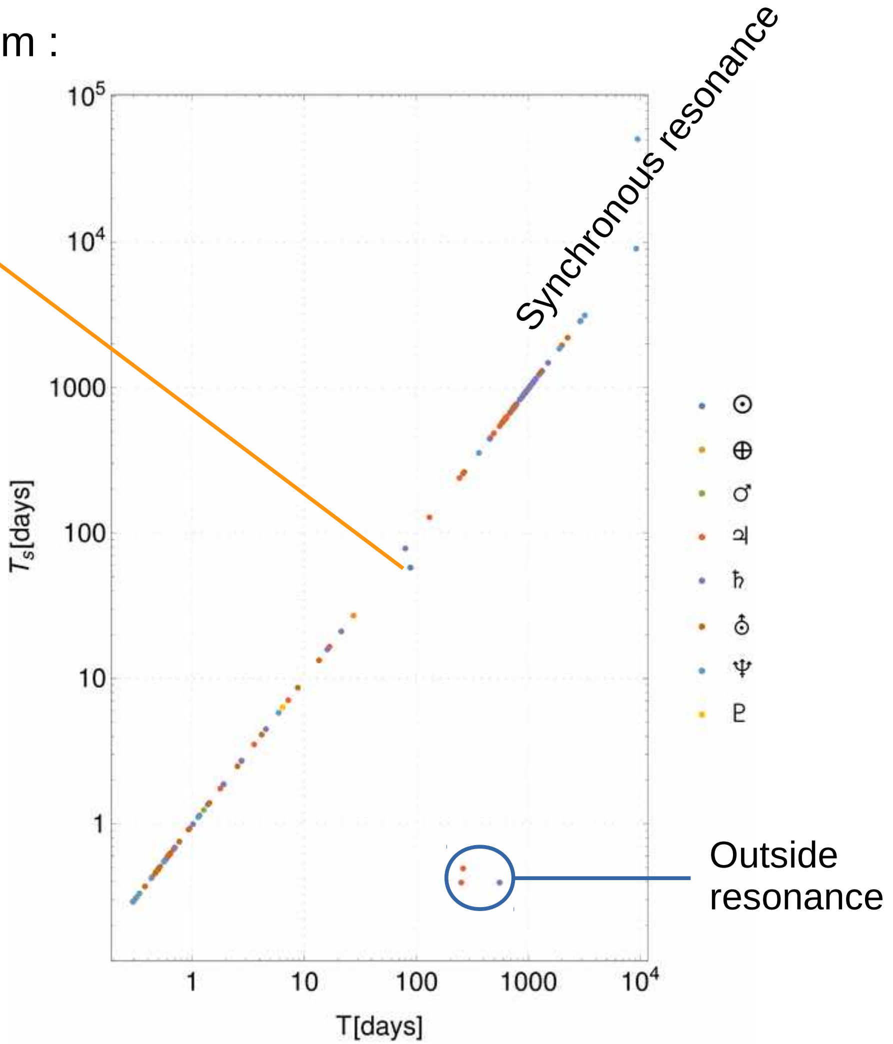
Image credits, see Wikimedia Commons

ÖAW IWF

Realizations in the solar system II

Rotation and orbital periods in the solar system :

Central Body	T/T_s	T	Body
Sun	3/2	87.9	Mercury
Earth	1	27.3	Moon
Mars	1	1.2	Deimos
Mars	1	0.3	Phobos
Jupiter	1	0.3	Adrastea
Jupiter	1	730.1	Aitne
Jupiter	1	0.5	Amalthea
Saturn	1	1116.5	Aegir
Saturn	1	783.5	Albiorix
Saturn	1	0.6	Atlas
Uranus	1	2.5	Ariel
Uranus	1	0.6	Belinda
Uranus	1	0.4	Bianca
Neptune	1	0.3	Despina
Neptune	1	0.4	Galatea
Neptune	1	1879.7	Halimede
Pluto	1	6.4	Charon

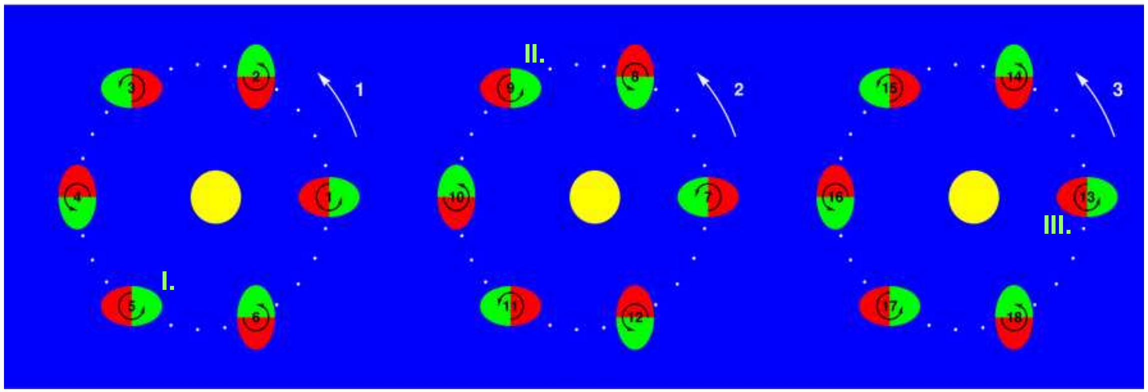


161 moons and 1 planet based on Wolfram Curated Data

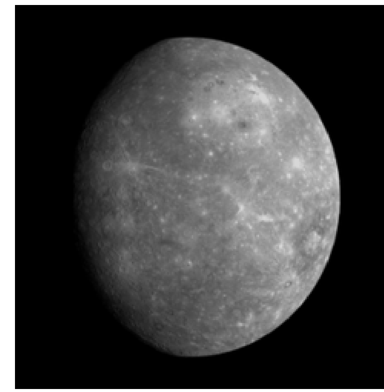
ÖAW IWF

Case study : planet Mercury

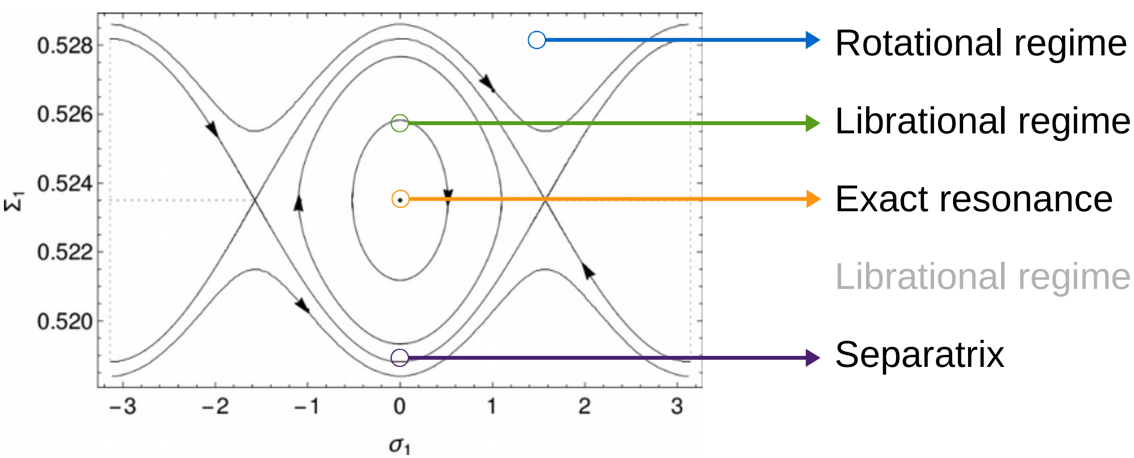
Sketch of two-body problem in 3:2 spin-orbit resonance :



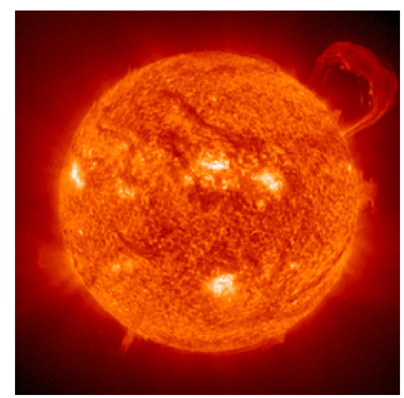
Planet Mercury :



Phase portrait in 3:2 resonant variables (Σ_1 σ_1) :



around the Sun :



ÖAW IWF

Parameters used in our study

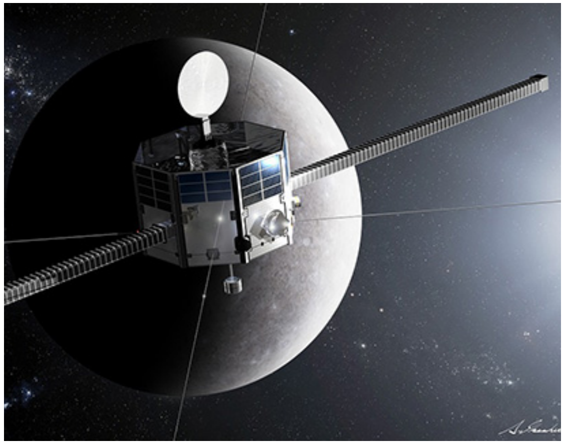
Quantity	Value
a	0.387098au
e	0.2056
*i	8.629 °
n	4.09235 °/d
P_{Ω}	329.162 ky
P_{ω}	134.477 ky
M	1 M_{sun}
m	0.055 M_{ear}
R	2440 km
c	0.349
**C_{20}	-50.32×10^{-6}
C_{22}	8.03×10^{-6}
C_{40}	-19.5×10^{-6}

*Inclination wrt Laplace plane

**Denormalized Stoke coefficients

Messenger mission (NASA)

Bepi Colombo (ESA, JAXA)



- Orbital elements are taken from Stark et al. 2015
- Physical parameters (gravity field) are taken from Mazarico et al 2014, Smith et al. 2012

We assume rigid body rotation

- Ok for latitudinal oscillations : the molten layer is viscous enough to act as a rigid one (Peale, 1976)
- For longitudinal librations the model needs to be refined

Rotational history of Mercury

Short review of the state of the art :

- Strong influence due to the planetary perturbations (D'Hoedt & Lemaitre, 2008)
- Spin-state is influenced by orbital dynamics, shape, and tides (Noyelles & Lhotka 2013, Baland et al., 2017)

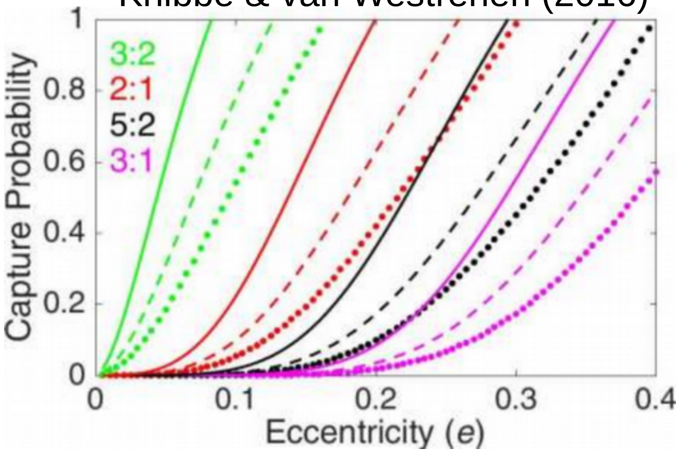
- Mercury's mantle is partly decoupled from the core that is partly molten (Margot et al. 2007)
- The size of the core is still unknown, thus the moment of inertia is still uncertain (Peale et al., 2016)
- Size of the inner core prolongs the long-period librations around exact resonance (Yseboodt et al. 2013)
- Accurate knowledge of the tide generating potential allows to constrain the size of the core (Van Hoolst et al., 2003, 2007)

- Probability of capture in either one of the 5:2, 2:1, 3:2 resonances can be increased due to core-mantle friction (Correia & Laskar, 2009)
- Possibility of capture can also be explained by more accurate tidal models (Noyelles et al. 2014)
- Initial retrograde rotation could also allow synchronous 1:1 spin-orbit resonance (Wiezcoreck et al. 2012)

ÖAW IWF

Probability of capture in p:q resonance

Knibbe & van Westrenen (2016)



Probability of capture of a Mercury-like terrestrial planet in 3:2, 2:1, and 5:2 spin-orbit resonances.

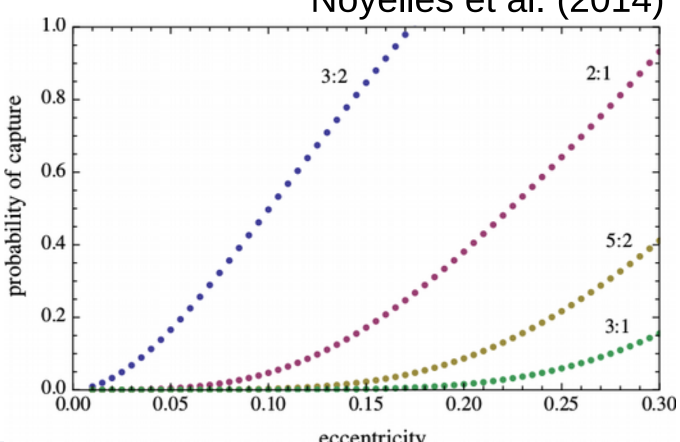
Correia & Laskar (2004)

$$P_{1/1} = 2.2\%, \quad P_{3/2} = 55.4\%, \quad P_{2/1} = 3.6\%$$

Capture probability at 3:2 (green), 2:1 (red), 5:2 (black), and 3:1 (purple) spin-orbit resonances for different triaxiality parameters (thick, dashed, dotted)

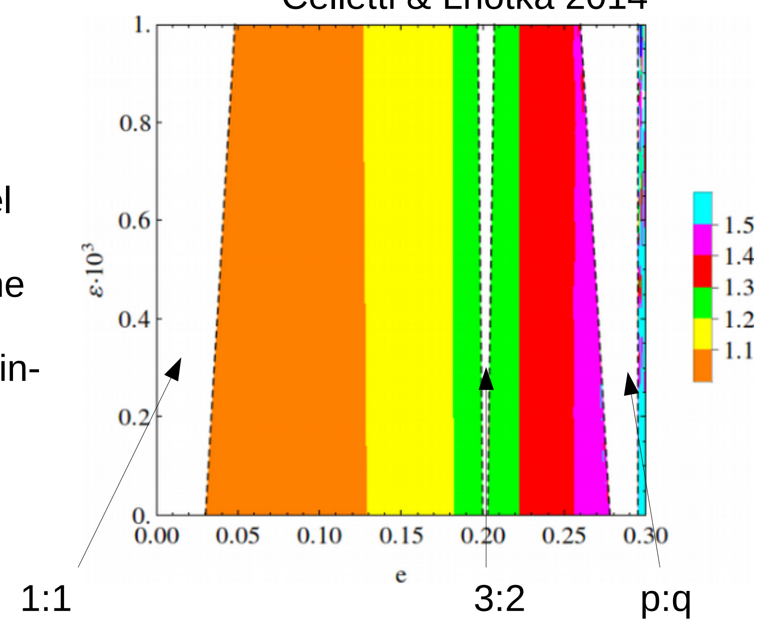
➔ Capture propability depends on orbital eccentricity and triaxility :

Noyelles et al. (2014)



Simple tidal model reveals the dependency on the parameters of the attractor of the spin-orbit problem :

Celletti & Lhotka 2014

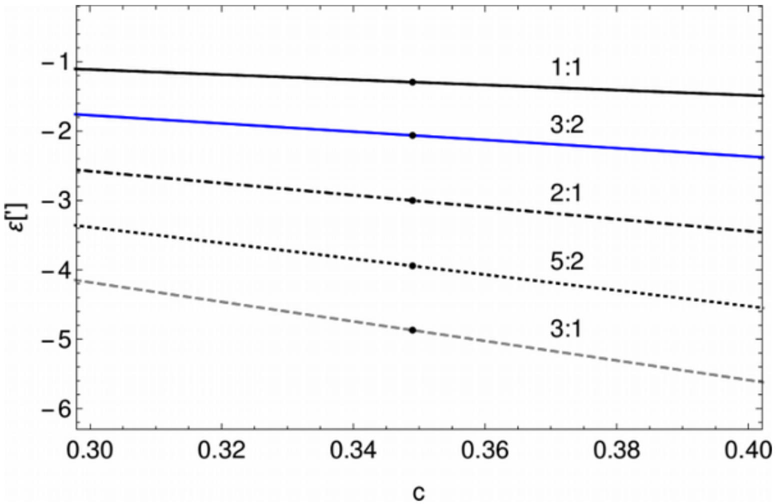


Used a new realistic tidal model

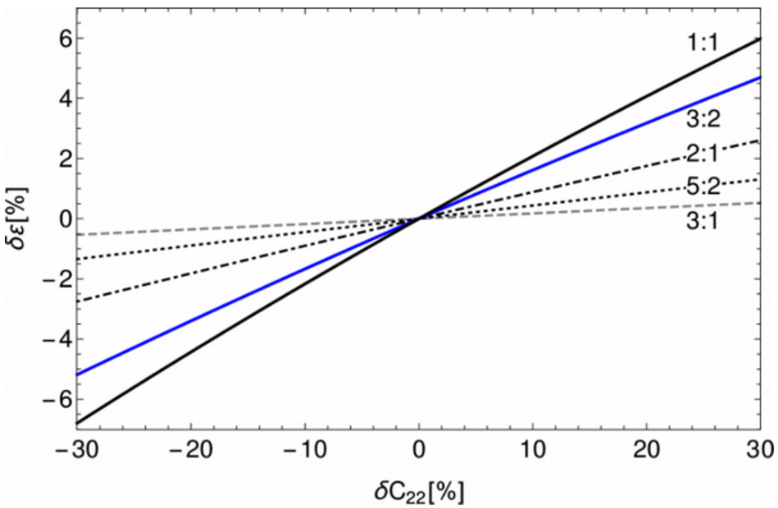
ÖAW IWF

Application to the obliquity of Mercury

Nearly linear relation between obliquity and normalized polar moment of inertia :



Weak quadratic relation between obliquity and equatorial flattening :



- Increasing magnitude of the obliquity for increasing polar moment of inertia
- Larger inclination between spin-axis and orbit normal for higher order resonances
- Current value of polar moment of inertia (0.349) gives: 1.29' for 1:1, 2.06 for 3:2, 3.00' for 2:1, 3.94 for 5:2, and 4.86' for 3:1 resonances

- Effect of C_{20} becomes stronger for lower order resonances
- 10% change in C_{20} gives about 1.66% change in obliquity for 3:2 resonance
- Change of 10% in C_{20} gives 2.16% in ε for 1:1, 1.00% for 2:1, 0.44% for 5:2 and 0.17% for 3:1 resonances

Lhotka, 2017

ÖAW IWF

Outlook : long term stability of motions

How stable are oscillations around exact equilibrium?

$$H_0 = \omega_{u_1} U_1 + \omega_{u_3} U_3$$

$$H^{(0)}(U,u) = \omega_u \cdot U + \sum_{j>0} H_j^{(0)}(U,u)$$

Fourier series expansion valid in the vicinity of the resonance

Here, H_j are homogeneous polynomials of degree $j/2 + 1$ in U

Birkhoff normal form, Giorgilli (1988)

How stable are oscillations around exact equilibrium?

$$H^{(r)}(U,u) = Z_0(U) + \ldots + Z_r(U) + \sum_{s>r} \mathcal{R}_s^{(r)}(U,u)$$

normalization order   normal for terms   remainder

Dynamical problem transformed to normal form coordinates

$$\Delta_{\varrho R} = \{U \in \mathbb{R}^2 : |U_j| \le \varrho R_j \ , \ j = 1,3\}$$

Initial conditions U (in the vicinity of the exact resonance) bounded by constants R_j

Bound on the possible evolution for finite times :

$$|f(U,u)| \le |f|_R \, \varrho^{s/2+1} \ , \quad \text{for } U \in \Delta_{\varrho R} \, , \ u \in \mathbb{T}^2$$

f some function that bounds effect of the remainder

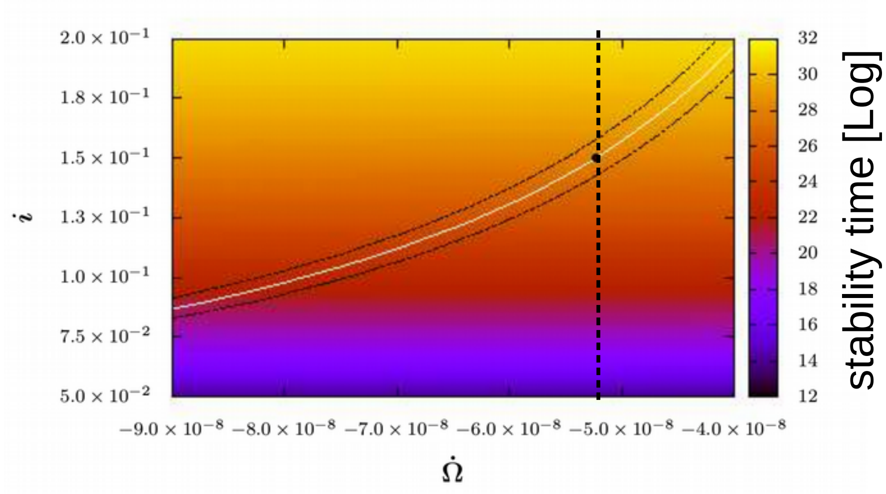
Nekhoroshev like stability estimates

Sansottera et al., 2015

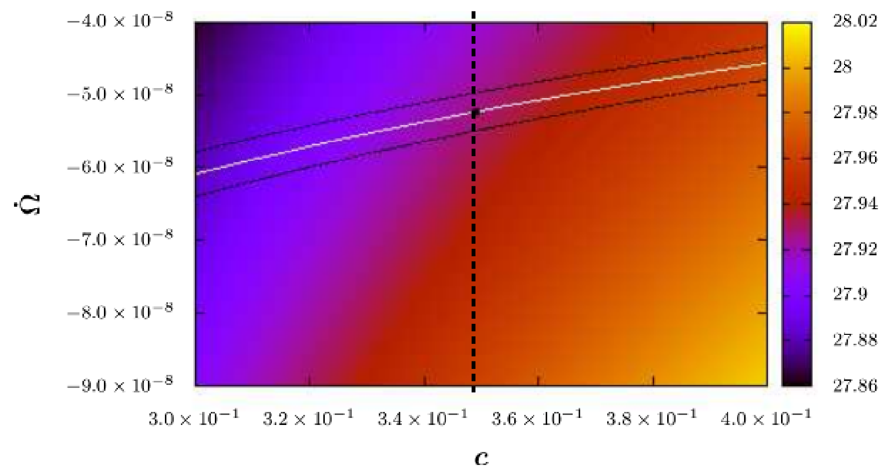
ÖAW IWF

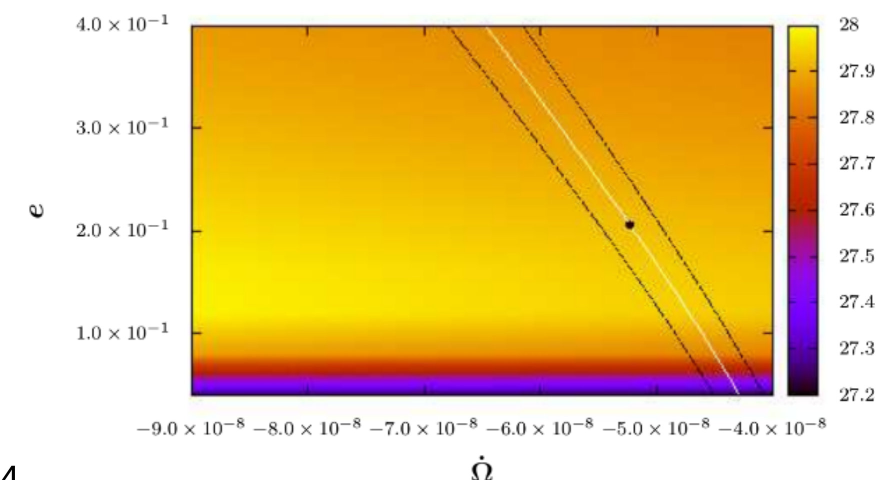
Outlook : long term stability of motions

Parametric study (for the 3:2 resonance) :



Today's position of Mercury





- The stability time is mostly influenced by the inclination
- Only moderate influenced by changes in :
  - polar moment of inertia
  - regression rate of the ascending node longitude
  - Eccentricity
- Weakly influenced by precesion rate of the perihelion

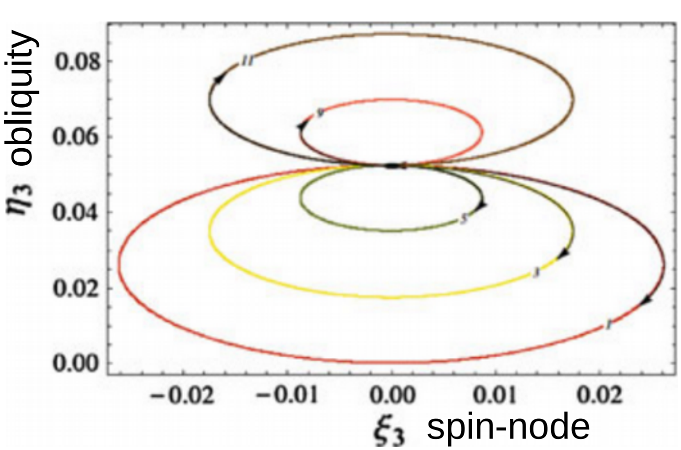
similar study for 1:1 resonance in Sansottera et al. 2014

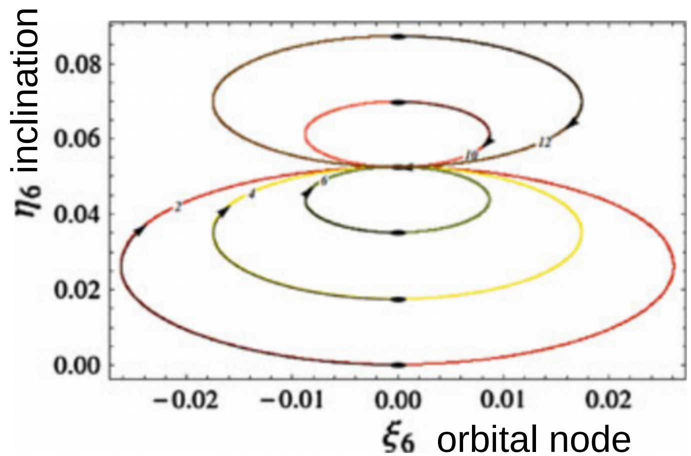
Sansottera et al., 2015

ÖAW IWF

Outlook : symplectic mapping

Long-term simulations based on symplectic integration :





Reduction of a 12-dimensional phase space problem

Spin-orbit and orbit-spin coupling effects

Long-term stability validated

Hadjidemetriou mapping :

$$\Sigma_1{}^{(n)} = \frac{\partial S''}{\partial \sigma_1{}^{(n)}}, \quad \sigma_1{}^{(n+1)} = \frac{\partial S''}{\partial \Sigma_1{}^{(n+1)}}$$

$$\eta_k{}^{(n)} = \frac{\partial S''}{\partial \xi_k{}^{(n)}}, \quad \xi_k{}^{(n+1)} = \frac{\partial S''}{\partial \eta_k{}^{(n+1)}}$$

- Resonant (semi-fast) motion
- Slow motion

Exists just for the 1:1 spin-orbit resonance

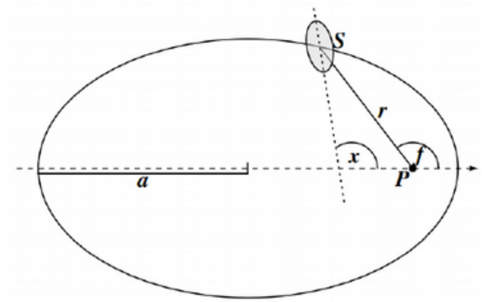
- Cassini states formulated in terms of fixed point solutions of the mapping
- Importance of coupling of the obliquity and inclination of the rotating body
- Linear stability analysis of the transfer matrix

Lhotka, 2013

ÖAW IWF

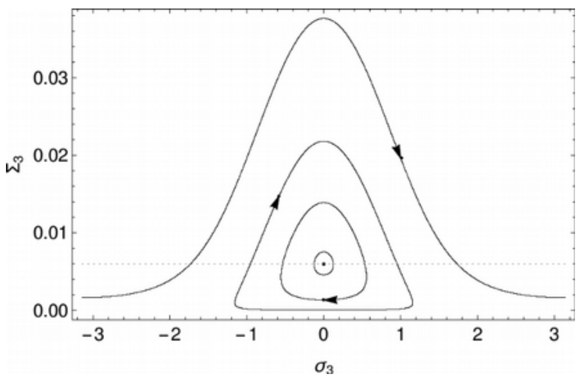
Summary and conclusions

- Cassini states and the spin-orbit problem
  - Generalization to p:q resonances
  - Study in higher degree gravity field harmonics

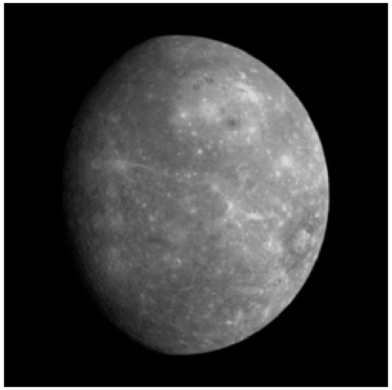


- Gravitational coupling
  - Degree and order 4
  - Isolation of resonant terms

$$\mathcal{V} = -\frac{\mathcal{GM}}{r}\sum_{n=2}^{\infty}\sum_{m=0}^{n}\left(\frac{R}{r}\right)^{n} P_{nm}(\sin(\phi))\,\left(C_{nm}\cos(m\lambda)+S_{nm}\sin(m\lambda)\right)$$



- Equilibria of the system
  - Generalized Peale's alike formulae
  - Theory of libration periods
- Application to Mercury
  - Study in polar moment of inertia
  - Role of the flattening coefficient



- Outlook
  - Long-term stability
  - Symplectic Hadjidemetriou mapping